# ON THE OSCILLATION OF SOLUTIONS OF CERTAIN SELF-ADJOINT LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER



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## CHAPTER I

## THE CLASS F

### I. INTRODUCTION

This paper investigates the oscillatory properties of solutions to certain self-adjoint linear differential equations of the fourth order. This topic has been studied by Barrett [1,2]*, Leighton and Nehari [4], and others. It is shown that the results of section three of [4] can be obtained under weaker hypothesis than assumed there. In at least one place new results, Lemma 3.4, were needed, but usually the techniques used in [4] still work under the weakened hypothesis.

A general self-adjoint linear differential equation of the fourth order is of the form

$$[r(x)y''(x)]'' + [q(x)y'(x)]' + p(x)y(x) = 0 \tag{1.1}$$

where $r$ and $p$ are continuous and $r(x) > 0$ for all $x$ on the interval under consideration and $q$ has a continuous derivative.

Before continuing it will be necessary to make some fundamental definitions.

DEFINITION 1.1. <u>A solution to a differential equation</u> (1.1) <u>is said to be oscillatory on the interval</u> $(a,\infty)$, <u>where</u> $a > 0$, <u>if it has an infinite number of zeros on this interval.</u>

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*The numbers which appear in brackets correspond to the Bibliography of this paper.

DEFINITION 1.2. *A solution to a differential equation (1.1) is said to be nonoscillatory on $(a, \infty)$, $a > 0$, if it has only a finite number of zeros on the interval.*

DEFINITION 1.3. *A differential equation is said to be oscillatory if at least one of its solutions is oscillatory, and nonoscillatory if all of its solutions are nonoscillatory.*

DEFINITION 1.4. *A solution to a differential equation is said to have a zero of order $k$ at a point $b > 0$ if the solution and its first $k - 1$ derivatives are zero at $b$. In theorems concerning the number of zeros of a solution, a zero of order $k$ will be counted as $k$ zeros.*

In sections three and four this paper will concern itself with a particular class of the differential equations of the type (1.1). This class will be denoted by $F$ and will consist of all equations of the form (1.1) with the property that if $y$ is a solution of the equation and $y(b) = 0 = y'(b)$, $b > 0$, then $y(x) \neq 0$ for all $x$ in $(0,b)$ or for all $x$ in $(b, \infty)$; moreover, if $y(b) = y'(b) = y''(b) = 0$ then $y(x) \neq 0$ for $x \neq b$.

In the last chapter of this paper particular nonempty subclasses of $F$ will be examined.

## II. PRELIMINARY RESULTS

We now consider some properties of general fourth order linear differential equations.

THEOREM 2.1. *Given three, not necessarily distinct, positive* *numbers, there exists a solution to* (1.1) *having zeros at these three numbers.*

PROOF. If all three zeros coincide then the result follows from the existence theorem [4]. Otherwise let $y_1$, $y_2$, $y_3$, and $y_4$ be linearly independent solutions of (1.1) and let $a_1$, $a_2$, $a_3$, and $a_4$ be solutions of the system

$$\sum_{i=1}^{4} a_i y_i(x_j) = 0 , \qquad (j = 1, 2, 3) ,$$

or the system

$$\sum_{i=1}^{4} a_i y_i^{(k)}(x_1) = 0 , \qquad (k = 0, 1) ,$$

$$\sum_{i=1}^{4} a_i y_i(x_2) = 0 ,$$

according as the three points are distinct or two of them coincide. Each system is a homogeneous system of three equations in four unknowns and a nontrivial solution always exists. Then the function $y$ defined by $y(x) = \sum_{i=1}^{4} a_i y_i(x)$ is a solution having the three desired zeros.

LEMMA 2.2. *Let* $u$ *and* $v$ *be of class* $C'$ *in* $(a,b)$ *and let* $v$ *have no zeros in this interval. If* $u$ *has two distinct zeros, say* $a_1$ *and* $a_2$ *in* $(a,b)$, *then the function* $S(x) = v(x)u'(x) = u(x)v'(x)$ *must* have a zero in $(a_1, a_2)$.

PROOF. Let $g(x) = u(x)/v(x)$ . Then $g'(x) = [v(x)u'(x) - u(x)v'(x)]/[v(x)]^{-2} = S(x)[v(x)]^{-2}$. So $\int_{a_1}^{a_2} g'(x)dx = \int_{a_1}^{a_2} S(x)[v(x)]^{-2}dx$.

The integral on the left may be evaluated by the fundamental theorem of integral calculus. Since $g(a_1) = g(a_2) = 0$, then $\int_{a_1}^{a_2} S(x)[v(x)]^{-2}dx = 0$. This implies that $S$ have at least one zero in $(a_1, a_2)$.

THEOREM 2.3. Let $u$ and $v$ be of class $C'$ in $(a,b)$ and let $v$ have no zeros in this interval. If in $(a,b)$ $u$ has two distinct consecutive zeros, say $a_1$ and $a_2$ then there exists a constant $\mu$ such that the function $z$ defined by $z(x) = u(x) - \mu v(x)$ has a double zero in $(a_1, a_2)$.

PROOF. These conditions exactly fit Lemma 2.2, so $S$ must have a zero, say $x_0$, in $(a_1, a_2)$. If $\mu = u(x_0)/v(x_0)$, then $z(x_0) = z'(x_0) = 0$; that is, $z$ has a double zero at $x_0$.

## III. THE NUMBER OF ZEROS OF A SOLUTION ON AN INTERVAL

This section concerns itself with the number of zeros of solutions to equations in $F$. One of the most important results of this section concerns two solutions to an equation in $F$ that have three zeros in common.

THEOREM 3.1. If $y_1$ and $y_2$ are two nontrivial solutions to an equation in $F$ that have three, not necessarily distinct, zeros in common, then $y_1$ and $y_2$ are constant multiples of each other.

PROOF. Let the three zeros be $a$, $b$, and $c$ where $0 < a \leq b \leq c$. The different configurations of zeros will have to be studied in cases.

Case I. All three zeros are distinct, $a < b < c$. Since there are zeros at $a$ and $c$ then $y_1'(b) \neq 0$ and $y_2'(b) \neq 0$. Define a function $w_1$ such that $w_1(x) = y_1'(b)y_2(x) - y_2'(b)y_1(x)$. Now $w_1$ is a solution to the equation under consideration since it is a linear combination of $y_1$ and $y_2$. Notice that $w_1(a) = w_1(b) = w_1(c) = w_1'(b) = 0$. This is a contradiction to the fact that $w$ is a solution to an equation in $F$, unless $w_1$ is identically zero. In that case $y_1(x) = [y_1'(b)/y_2'(b)]y_2(x)$. So $y_1$ and $y_2$ are constant multiples.

Case II. Two of the zeros coincide. There are two such cases, but it will be sufficient to examine one of them. Let $a = b < c$. Therefore $y_1(b) = y_1'(b) = y_1(c) = 0$ and $y_2(b) = y_2'(b) = y_2(c) = 0$. If it were the case that $y_1''(b) = 0$ or that $y_2''(b) = 0$, then the equation under consideration could not be in $F$. So $y_1''(b) \neq 0$ and $y_2''(b) \neq 0$. Define a function $w_2$ by $w_2(x) = y_1''(b)y_2(x) - y_2''(b)y_1(x)$. Now $w_2$ is a solution to the equation under consideration, and $w_2(b) = w_2'(b) = w_2''(b) = w_2(c) = 0$. This is a contradiction to the fact that $w_2$ is a solution to an equation in $F$, unless $w_2$ is identically zero. In that case $y_1(x) = [y_1''(b)/y_2''(b)]y_2(x)$. So $y_1$ and $y_2$ are constant multiples.

Case III. All three zeros coincide, $a = b = c$. Therefore $y_1(b) = y_1'(b) = y_1''(b) = 0$ and $y_2(b) = y_2'(b) = y_2''(b) = 0$. In this case both $y_1$ and $y_2$ are multiples of the solution $y$ satisfying the conditions $y(b) = y'(b) = y''(b) = 0$ and $(ry'')'(b) = 1$. Hence $y_1$ and $y_2$ are constant multiples.

Thus in all cases $y_1$ and $y_2$ are constant multiples if they have three zeros in common.

Zeros of solutions to equations in $F$ cannot be expected to separate each other as do zeros of solutions to the second order self-adjoint equation. Since three zeros may be chosen and a solution to the equation in $F$ constructed to pass through the three zeros, then a solution could be constructed so that it would have three zeros between consecutive zeros of another solution. Simple separation of zeros of two solutions of an quation in $F$ is, however, possible if the right conditions are met by the two solutions as will be seen in the following theorem.

THEOREM 3.2. <u>If</u> $u$ <u>and</u> $v$ <u>are two linearly independent solutions of an equation in</u> $F$ <u>such that</u> $u(x) = v(a) = u(b) = v(b) = 0$, $0 < a < b < c$, <u>then the zeros of</u> $u$ <u>and</u> $v$ <u>in</u> $(a,b)$ <u>separate each other.</u>

PROOF. Let $u$ and $v$ both have zeros at $a$ and $b$, where $0 < a < b$. If the theorem is not true, then in the interval $(a,b)$ one of the solutions, say $u$, has two consecutive zeros, say $a_1$ and $a_2$, which are not separated by zeros of $v$. Let $w$ be a solution defined by $w(x) = u(v) - \mu v(x)$, where $\mu$ is a nonzero constant. Now $w(a) = w(b) = 0$ for all values $\mu$. By Lemma 2.3 there is a $\mu$ that will cause $w$ to have a double zero in $(a_1, a_2)$, at say $\gamma$. Now $w$ is a solution to an equation in $F$, and it has a double zero with single zeros on either side in $(a,b)$. This is a contradiction. Then it must be the case that the zeros of $u$ and $v$ separate on $(a,b)$.

This result leads directly to the following corollary.

COROLLARY 3.3. *If $u$ and $v$ are two nontrivial solutions of an equation in $F$ with zeros at $a$ and $b$ in common, then the number of zeros of $u$ in $(a,b)$ differs from the number of zeros of $v$ in $(a,b)$ by at most one.*

PROOF. Note that neither $u$ nor $v$ may have any multiple zeros in $(a,b)$. If one of the solutions, say $u$, had two or more zeros in excess of the number that the other, $v$, had in $(a,b)$, then $u$ would have two consecutive zeros not separated by $v$. By Theorem 3.2 the zeros of $u$ and $v$ must separate, which is a contradiction. Thus the number of zeros of $u$ and $v$ in $(a,b)$ may differ by at most one.

The following result is useful before reducing the restrictions of Theorem 3.2.

LEMMA 3.4. *If $u$ and $v$ are two nontrivial solutions of an equation in $F$ and either the zeros of $u$ separate the zeros of $v$ in $(a,a_k)$ or $u$ and $v$ have a zero in common in $(a, a_k)$, for $0 < a < a_k$, but $v$ does not separate the two consecutive zeros, $a_k$ and $a_{k+1}$ of $v$, then the zeros of $u$ do not separate the zeros of $v$ in $(a_{k+1}, \infty)$. Further, $u$ and $v$ have no zeros in common in $(a_{k+1}, \infty)$.*

PROOF. Each time the zeros of $u$ and $v$ separate each other, the linear combination $z(x) = u(x) - \mu v(x)$, where $\mu$ is a nonzero constant, has a single zero in the open interval bounded by consecutive zeros of one solution which are separated by a zero of the other solution. Now if the zeros of $u$ and $v$ separate one or more times in $(a_{k+1}, \infty)$, then $z$ has a single zero in $(a,a_k)$ and $(a_{k+1},\infty)$. However, $u$ is nonzero in $(a_k, a_{k+1})$. By Lemma 2.3 there is a $\mu$, such that $z$ has a double zero in $(a, a_{k+1})$, but this contradicts the fact that $z$ is a solution to an equation in $F$. Then it must be the case that $z$ has no single zeros in $(a_{k+1}, \infty)$. But $z$ has single zeros in $(a_{k+1}, \infty)$ if $u$ and $v$ separate in the interval or if they have a zero in common. Hence, $u$ and $v$ may not separate or have zeros in common in $(a_{k+1}, \infty)$.

The following lemma results from the same argument.

LEMMA 3.4'. If the zeros of $u$ and $v$ separate or if $u$ and $v$ have a zero in common in $(b_{k+1}, b)$ and $v(x) > 0$ in $[b_k, b_{k+1}]$, where $b_k$ and $b_{k+1}$ are consecutive zeros of $v$, then the zeros of $u$ and $v$ do not separate in $(0, b_k)$. Further $u$ and $v$ have no common zeros in $(0, b_k)$.

These lemmas are used in the proof of the following theorem.

THEOREM 3.5. Let $u$ and $v$ be two nontrivial solutions of an equation in $F$ which vanish at some $a > 0$. If $u(b) = v(c) = 0$, $a < b < c$, and $n$ and $m$ denote the number of zeros of $u$ and $v$, respectively, in $(a, b]$, then $n - 2 \leq m \leq n + 2$.

PROOF. Without restriction let $c$ be the smallest zero of $v$ in $(b, \infty)$. If $v(b) = 0$, then $u$ and $v$ have two zeros in common, and by Corollary 3.3 $u$ and $v$ differ by at most one zero on $(a, b)$. Now either $u$ and $v$ both have a single zero at $b$ or $u$ has a double zero and $v$ has a single zero. In either case $n - 2 \leq m \leq n + 2$. If $v(b) \neq 0$ then there are two cases to be considered.

Case I. In this case $u$ has a single zero at $b$ and $v$ has no zero at $b$. Let $w(x)$ be a solution to the equation under consideration with the three zeros $a$, $b$, and $c$. By Theorem 3.3 the zeros $u$

and $w$ on $(a,b)$ differ by at most one, but $u$ has a single zero at $v$ as does $w$. Then the zeros of $u$ and $w$ on $(a,b]$ differ by at most one. Likewise, the zeros of $v$ and $w$ separate on $(a,c)$ and must therefore separate on $(a,b]$. If the zeros of $v$ and $w$ on $(a,b]$ separate, then they differ by at most one. Then the zeros of $u$ and $v$ on $(a,b]$ can differ by at most two. Hence $n - 2 \quad m \quad n + 2$.

Case II. Here $u$ has a double zero at $b$, and $v$ has no zero at $b$. From the argument in Case I it can be seen that $n - 3 \leq m \leq n + 2$, since $u$ has one additional zero at $b$. If $u$ actually has three more zeros than $v$ in $(a,b]$, then $u$ has one more zero than $v$ in $(a,b)$. Now assume that $u''(b) > 0$ and $v(b) > 0$. Neither $u$ nor $v$ may have any multiple zeros in $(a,b)$. If $a$ and $b$ are two consecutive zeros of $u$, where $a < b$, then by Lemma 3.4 $v$ may not have exactly one zero in $(a_\ell, b)$. If $v$ has no zero in $(a_\ell, b)$, then the linear combination $z(x) = u(x) - \mu v(x)$ has a double zero in $(a_\ell, b)$ for some $\mu > 0$, and a single zero at $a$. Further, $z$ has a zero in $(b,c)$. This is not possible as $z$ is a solution to an equation in $F$. If $v$ has two or more zeros in $(a_\ell, b)$ and if the number of zeros of $u$ exceeds the number of zeros of $v$ by three on $(a,b]$, then the number of zeros of $u$ exceeds the number of zeros of $v$ by three or more on $(a, a_\ell]$. It was seen in Case I that the number of zeros of $u$ and $v$ on $(a, a_\ell]$ differs by at most two. Then the assumption that $u$ had three more zeros than $v$ in $(a,b]$ is false. Hence $n - 2 \leq m \leq n + 2$.

As a result of this theorem the following corollary is immediate.

COROLLARY 3.6. If $u$ and $v$ are two nontrivial solutions to an equation in $F$ where $u(a) = v(a) = u(b) = v(c) = 0$, $0 < a < b < c$, and $u$ has a double zero at $b$ and at least one other zero in $(a,b)$; then $v$ must vanish in $(a,b]$.

The restrictions of Theorem 3.2 were reduced to form Theorem 3.5. Likewise, the restrictions will now be further reduced to form Theorem 3.7.

THEOREM 3.7. Let $u$ and $v$ be two nontrivial solutions to an equation in $F$ for which $u(a) = u(b) = v(a') = v(b') = 0$, $0 < a' < a < b < b'$. If $n$ and $m$ denote the number of zeros in $[a,b]$ of $u$ and $v$, respectively, then $n - 3 \leq m \leq n + 3$.

PROOF. If $v(a) = 0$ and $q$ and $p$ are the number of zeros of $u$ and $v$, respectively, on $(a,b]$, then by Theorem 3.5 $q - 2 \leq p \leq q + 2$. Now $v$ may have only a single zero at $a$ under these conditions, and $u$ may have a single or a double zero. In either case $n - 3 \leq m \leq n + 3$. If $v(a) \neq 0$, then let $w$ be a solution to the equation under consideration with zeros at $a'$, $a$, and $b'$. Then the zeros of $w$ and $v$ on $[a,b]$ separate and their number differs by at most one. Further $u$ and $w$ satisfy Theorem 3.5 on $(a,b]$ and their zeros on this interval may therefore differ by at most two. It must be the case that $w$ has exactly one zero at $a$. If $u$ has a single zero at $a$, then the number of zeros of $u$ and $w$ on $[a,b]$ differ by at most two. This being the case, then $n - 3 \leq m \leq n + 3$. Then in the only case left to be examined $u$ has a double zero at $a$ and $v$ has no zero at $a$. The previous argument shows that $n - 4 \leq m \leq n + 3$ in this case. Assume without loss of generality that $v(a) < 0$. Let $a_\ell$ and $a_1$ be consecutive zeros of $u$, $a < a_1$, and $a_\ell$ and $b$ be consecutive zeros of $u$, $a < b$. By Lemma 3.4 and 3.4', $v$ cannot have exactly one zero in both $(a, a_1)$ and $(a\ ,b)$. Then for the proper choice of $\mu$ the linear combination $z(x) = u(x) - \mu v(x)$ has a double zero in one of these intervals.

If $z$ has its double zero in $(a, a_1)$, then either $v$ has no zeros in $(a, a_1)$ or more than one. If $v$ has no zeros in $(a, a_1)$, then $z$ has a zero in $(a', a)$, a double zero in $(a, a_1)$, and at least one more zero in $(a_1, b')$. This is not possible, as $z$ is a solution to an equation in $F$. If $v$ has two or more zeros in $(a, a_1)$ and $n - 4 = m$, then $u$ must have at least four more zeros than $v$ on $(a, b]$. Now let $w_1$ be a solution to the equation under consideration with zeros at $a'$, $a_1$, and $b'$. Then $w_1$ and $v$ differ by at most one zero on $(a_1, b]$ and $w_1$ and $u$ differ by at most two zeros on $(a_1, b]$. So, $u$ and $v$ will differ by at most three zeros on $(a_1, b]$, which is a contradiction. The case where $z$ has its double zero in $(a\ ,b)$ may be handled by a similar argument.

## IV. CONJUGATE POINTS

An important concept in the study of the oscillation of solutions to equations in $F$ is that of the $n$-th conjugate point.

DEFINITION 4.1. Let $y$ be a solution of an equation in $F$ and suppose $y$ has at least $n + 3$ zeros $a_1, a_2, \ldots, a_{n+3}$ $(a = a_1 \leq a_2 \leq \cdots \leq a_{n+3}, n \geq 1)$ for $x \geq a$. The $n$-th conjugate point of $a$ is the minimum value of $a_{n+3}$ as $y$ ranges over all solutions of the equation where $y(a) = 0$. It will be written as $\eta_n(a)$.

It will be seen in Theorem 4.1 that the extremal solution $y$ desired, if in fact one exists, must have a double zero at $a$, since any other solution with a single zero at $a$ and a zero at $a_{n+3}$ may have at most as many zeros as $y$ on $(a, a_{n+3})$.

THEOREM 4.1. *If there exists a solution $y$ of an equation in $F$ which vanishes at $a$ and has at least $n + 3$ zeros in $[a, \infty)$ then there exist $n$ points $\eta_1, \ldots, \eta_n$ $(a < \eta_1 < \eta_2 < \ldots < \eta_n)$ and $n$ pair wise linearly independent solutions $y_1, \ldots, y_n$ of the equation under consideration with the following properties:*

(a) *for each $p$ from 1 to $n$ the function $y_p$ has double zeros at $a$ and $\eta_p$;*

(b) *$y_p$ has precisely $p + 3$ zeros in $[a, \eta_p]$;*

(c) *any other solution $z$, where $z(a) = 0$ has fewer than $p + 3$ zeros in $[a, \eta_p]$.*

PROOF. Let $v$ be a solution to the equation under consideration satisfying the conditions $v(a) = v'(a) = v(a_{n+3}) = 0$, $v''(a) > 0$, and $v$ has its first $n + 3$ zeros at $a_1, \ldots, a_{n+3}$ $(a_n = a_1 = a_2 < a_3 < \ldots < a_{n+2} \leq a_{n+3})$. Likewise let $w$ be a solution to the same equation satisfying $w(a) = w'(a) = w''(a) = 0$ and $[rw'']'(a) = 1$, then $w(x) > 0$ for all $x$ in $(a, \infty)$. By Lemma 2.3 there is a constant $\mu_p \neq 0$ that causes $y_p$, defined by

$$(4.1) \qquad y_p(x) = v(x) - \mu_p w(x) ,$$

to have a double zero, at say $\eta_p$, in $(a_{p+2}, a_{p+3})$, if $a_{p+2}$ and $a_{p+3}$ are distinct. If $a_{p+2} = a_{p+3}$, then this gives $v$ a double zero at $a_{p+3}$. Since $v$ already had a double zero at $a$ it may not have any other zeros in $(a_{p+3}, a_{n+3})$. Then this may only happen if $a_{p+3} = a_{n+3}$, that is $n = p$.

If $a_{n+2} = a_{n+3}$, then $v$ has a double zero at $a_1$, and at $a_{n+3} = \eta_n$, therefore let $\mu_n = 0$ in this case. So $y_p$, given by (4.1), has double zeros at $a$ and $\eta_p$. This completes part (a) of the proof.

Assume that $v(x) \geq 0$ for $x$ in $(a_{p+2}, a_{p+3})$ if $a_{p+2} \neq a_{p+3}$. Then $\mu_p > 0$. It must now be the case that $\mu_p w$ intersects each positive arch of the curve $v$ in $(a_3, a_{p+2})$ at exactly two distinct points. If $\mu_p w$ did not intersect an arch of $v$ in $(a_3, a_{p+2})$, then by Lemma 2.3 there would be a $\mu_p^*$ such that $v - \mu_p^* w$ would have a double zero, at say $\gamma_1$, in the interval where $\mu_p w$ did not intersect a positive arch of $v$.

Now it is the case that $0 < \mu_p^* < \mu_p$, since $\mu_p w > v$ on the interval and $\mu_p^* w(\gamma_1) = v(\gamma_1)$. Hence $0 < \mu_p^* < \mu_p$. By a similar argument if $\mu_p w$ intersects an arch of $v$ more than twice, there is a $\mu_p^*$, $0 < \mu_p^* < \mu_p$, that causes $v - \mu_p^* w$ to have a double zero in the interval. Since $0 < \mu_p^* < \mu_p$, then $\mu_p^* w$ must intersect $v$ in the interval $(a_{p+2}, a_{p+3})$. This situation gives the solution $v(x) - \mu_p^* w(x)$, to the differential equation under consideration, double zeros at $a$ and $\gamma_1$, and a single zero at some $\gamma_2$ in $(a_{p+2}, a_{p+3})$ where $0 < a < \gamma_1 < \gamma_2$.

But this contradicts the fact that the differential equation is in $F$. So $\mu_p w$ intersects each positive arch of $v$ in $(a_3, a_{p+2})$ exactly two distinct points. Notice also that $y_p$ may have only single zeros in $(a, \eta_p)$. Then $y_p$ has twice as many zeros on $(a_3, a_{p+2})$ as there are positive arches of $v$ on the interval. Since $v$ was assumed positive on $(a_{p+2}, a_{p+3})$, then $y_p$ has $p - 2$ zeros on $(a_3, a_{p+2})$. It must be the case that $y_p$ has no zeros in $(a_{p+2}, \eta_p)$. Otherwise $y_p$ will have a double zero at $\eta_p$, a zero in $(\eta_p, a_{p+3})$, and a zero at $a$. This may not happen as the differential equation is in $F$. Now $y_p$ has exactly one zero in $(a_2, a_3)$. Notice that $v$ and $w$ are positive in $(a_2, a_3)$, and $v(a_3) = 0$.

If $\mu_p w$ did not intersect $v$ in $(a_2, a_3)$, then $rw''(a) > rv''(a)$, but $rw''(a) = 0$ and $rv''(a) > 0$. If $\mu_p w$ intersected $v$ more than once in $(a_2, a_3)$, then there is a $\mu_p^{**}$, $0 < \mu_p^{**} < \mu_p$, such that $v - \mu_p^{**} w$ has a double zero in $(a_2, a_3)$, a double zero at $a$, and a single zero to the right of $a_3$. If it is the case that $a_{p+2} = a_{p+3}$, then as noted before $y_p = v$. Then in either case $y_p$ has $p + 3$ zeros in $[a, \eta_p]$. The result is the same if $v(x) < 0$ for $x$ in $(a_{p+2}, a_{p+3})$. This proves the (b) part of the theorem.

Let $z_p$ be a solution to the same differential equation, but linearly independent of $y_p$, where $z_p(a) = z_p(\eta_p) = 0$. Then $z_p'(a) \neq 0$, otherwise $z_p$ and $y_p$ share three zeros which by Theorem 3.1 causes them to be linearly dependent. Now, $y_p$ and $z_p$ share zeros at $a$ and $\eta_p$ and hence separate on $(a, \eta_p)$. By Corollary 3.3 the number of zeros of $y_p$ and $z_p$ on $(a, \eta_p)$ may differ by at most one. Assume that $z_p$ has one more zero than $y_p$ on $(a, \eta_p)$. Since $y_p$ has double zeros at $a$ and $\eta_p$ and $z_p$ has only single zeros, then $y_p$ must have at least one more zero on $[a, \eta_p]$ than any other solution that passes through the points. This proves the final part of the theorem.

Thus $y_p$ is the extremal solution desired and the $\eta_p$'s are in fact the p-th conjugate points of Definition 4.1.

In the proof of Theorem 4.1 the points $\eta_p$ separated the zeros of $v$ on $(a, \infty)$, unless possibly $a_{n+2} = a_{n+3}$ in which case $\eta_n = a_{n+2}$. If the $y$ of Theorem 4.1 has a double zero at $a$, then $y$ is a constant multiple of $v$. This yields the following result.

THEOREM 4.2. <u>If $y$ is a solution to an equation in $F$ such that $y(a) = y'(a) = 0$, then the zeros of $y$ in $(a, \infty)$ are separated by the points $\eta_p(a)$ conjugate to $a$ (except when the last zero of $y$ is double in which case it coincides with a conjugate point).</u>

From Theorem 4.1 and Theorem 4.2 it would seem that the conjugate points are closely related to the oscillatory or nonoscillatory behavior of the equation. Theorem 4.3 and Theorem 4.4 show what the relationship is.

THEOREM 4.3. <u>An equation in $F$ is oscillatory if, and only if, for every $a > 0$ there exists an infinity of conjugate points $\eta_p(a)$.</u>

PROOF. By Definition 1.3 the equation in $F$ is oscillatory if, and only if, it has at least one oscillatory solution $y$. Since $y$ is oscillatory it must have an infinite number of zeros in $[a, \infty]$ for any $a > 0$. Suppose $a > 0$ and $n$ is any positive integer. Let $a_i$ be the i-th zero of $y$ to the right of $a$. Let $u$ be a solution satisfying $u(a) = u(a_1) = u(a_{n+4}) = 0$. By Theorem 3.3 $u$ will have at least $n + 3$

zeros in $[a, a_{n+4}]$ . By Theorem 4.1 there exists at least $n$ conjugate points of $a$ . Since $n$ was an arbitrary positive integer, then there will be an infinity of conjugate points of $a$ .

If there are an infinite number of conjugate points $\eta_p(a)$ then by Theorem 4.1 there are solutions to the differential equation with arbitrarily many zeros in $(a, \infty)$ . The extremal solutions $y_p$ shown in the proof of the theorem will meet these conditions. Now it is necessary to show that there is a solution with an infinity of zeros. By the existence and uniqueness theorem the solutions $y_p$ , may be written in the form $y^*(x) = c_1u(x) + c_2v(x)$, where $u$ and $v$ are two linearly independent $y_p$'s . Let $K$ be the class of functions $y^*$ given by

$$K = \left\{y \mid \exists\ p \ni y = (c_1{}^2 + c_2{}^2)^{-\frac{1}{2}} y_p \quad , \quad \text{where } y_p = c_1u + c_2v\right\} .$$

Likewise let

$$K_1 = \left\{y' \mid y \in K\right\}$$

$$K_2 = \left\{ry'' \mid y \in K\right\}$$

$$K_3 = \left\{(ry'')' \mid y \in K\right\}$$

$$K_4 = \left\{(ry'')'' \mid y \in K\right\} .$$

The $K$'s are equicontinuous and uniformly bounded classes. Then by applying Ascoli's Theorem [5] to subsets of $K$, $K_1$, $K_2$, $K_3$ , and $K_4$ a subsequence of $K$ is found that converges uniformly to a function $y_0$ , that is a solution to the differential equation under consideration. Any limit point of zeros of the $y_p$'s in the convergent subsequence will also be zeros of $y_o$. In Theorem 4.2 it was shown that all the $y_p$'s vanish in the interval $(\eta_{k-1}, \eta_k)$ if $k < p$. So $y_o$ has a zero in each of the intervals $[\eta_{k-1}, \eta_k]$.

This result may now be used to prove the following theorem.

THEOREM 4.4. *If an equation in F is nonoscillatory, then there exists an $a > 0$ such that no solution of the equation has more than three zeros in $(a, \infty)$.*

PROOF. If the theorem were not true, then it would be the case that for all $a > 0$ there exists a solution $y$, to the equation, where $y$ has at least four zeros in $(a, \infty)$. Then there exists a sequence of $a_i$'s $(0 < a_1 < a_2 < \dots)$ tending to infinity such that for each $a_i$ there is a solution $y_i$ of the equation which has four zeros in $(a_i, \infty)$, and $a_{i+1}$ is to the right of the fourth zero of $y_i$ in $(a_i, \infty)$. Let $v_i$ be a solution to the differential equation with a double zero at $a_1$ and another zero at $a_i$. By Theorem 3.7 $v_i$ must have a zero in each of the intervals $(a_p, a_{p+1})$, $(p = 1, 2, \dots, i - 1)$.

Since $i$ can be any positive integer, then there will exist solutions of the differential equation with a double zero at $a$ and arbitrarily many zeros on $(a_1, \infty)$. By Theorem 4.1, there are then an infinity of conjugate points of $a_1$. Therefore, by Theorem 4.3 the differential equation in F must be oscillatory which contradicts the original assumption. So the theorem follows.

## CHAPTER II

## CERTAIN EQUATIONS IN THE CLASS F

### I. THE EQUATION $[ry'']'' - py = 0$

The purpose of this section is to show that the equation

$$[ry'']'' - py = 0 \,, \tag{5.1}$$

where $r$ and $p$ are continuous on $(a_0, \infty)$, $a_0 > 0$, $r(x) > 0$, and $p(x) > 0$ for all $x$ in $(a_0, \infty)$, is in the class $F$. This was shown by Leighton and Nehari [[illegible]].

LEMMA 5.1. *If* $y$ *is a solution of* (5.1) *and the values of* $y$, $y'$, $y''$, *and* $(ry'')'$ *at* $a$, $a_0 < a$, *are all nonnegative and not all zero, then* $y$, $y'$, $y''$, *and* $(ry'')'$ *are all strictly positive for all* $x$ *in* $(a, \infty)$.

PROOF. The Taylor series expansion with integral remainder of the function $ry''$ about the point $a$ can be written as

$$ry'' = [ry'']_a + (x - a)[ry'']'_a + \int_a^x (x - t)p(t)y(t)dt \,. \tag{5.2}$$

Notice that everything to the right of the equality is nonnegative on the interval $(a, \infty)$ except possibly $y$. Then $ry''$ is positive whenever $y$ is positive, since the integral cannot be zero. So $y''$ must be positive if $y$ is positive, since $r$ is strictly positive. Now $y(a)$ is either positive or zero; if $y(a)$ is zero then the first nonzero quasi-derivative at $a$ is positive. In any case there is an interval $(a,x_o)$ in which $y$ is strictly positive. Suppose $y(x_o) = 0$. Now $y$ and $y'$ are positive or zero at $a$ and $y''$ remains positive as long as $y$ is nonnegative. Moreover, $y'$ must be positive as long as $y''$ is positive. But this implies that $y'(x) > 0$ for all $x$ in $(a,x_o]$. If $y$ is strictly positive in $(a,x_o)$ then $y'$ must be negative someplace in $(a,x_o)$ if $y(x_o) = 0$, but this is a contradiction. Then there cannot be a value of $x$ in $(a,\infty)$. If $y$ remains positive, then so do $y'$ and $y''$. By differentiation of (5.2),

$$[ry'']' = [ry'']'_a + \int_a^x p(t)y(t)dt \,, \tag{5.3}$$

$(ry'')'$ is shown ~~to remain~~ positive so long as $y$ does. Therefore $y, y'$, $y''$, and $(ry'')'$ are strictly positive for all $x$ in $(a,\infty)$.

LEMMA 5.1'. *If* $y$ *is a solution of* (5.1) *and the values of* $y$, $y'$, $y''$, *and* $(ry'')'$ *at* $a$, $a_o < a$, *are all nonpositive and not all zero, then* $y$, $y'$, $y''$, *and* $(ry'')'$ *are all strictly negative in* $(a,\infty)$.

PROOF. The proof follows directly from Lemma 5.1 if $y$ is replaced by $-y$. Note that $-y$ is a solution to (5.1) if $y$ is.

LEMMA 5.2. *If* $y$ *is a solution of* (5.1) *and if there is a* $c$ *in* $(a_o,\infty)$ *for which* $y(c) \geq 0$, $y'(c) \leq 0$, $y''(c) \geq 0$, *and* $[ry''] \leq 0$ *for all* $x$ *in* $(a_o,c)$; *then* $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0$, *and* $[ry'']' < 0$ for all $x$ in $(a_o,\infty)$.

PROOF. Let $z$ be a solution to (5.1) satisfying the conditions of Lemma 5.2. Let $y$ be a function given by $z(t) = y(a_o + c - t) = y(x)$, where $x = a_o + c - t$. Now $y(x) = z(t)$, $-y'(x) = z'(t)$, $y''(x) = z''(t)$,

$-[r(a_0+c-x)y''(x)]' = [r(t)z''(t)]'$, and $[r(a_0+c-x)y''(x)]'' = [r(t)z''(t)]''$. So $y$ satisfies the differential equation

$$(5.1') \qquad [r(a_0+c-x)y''(x)]'' - p(a_0+c-x)y(x) = 0$$

which is of the type (5.1). By supposition $z$ satisfies Lemma 5.2. Now $z(c) \geq 0$, $z'(c) \leq 0$, $z''(c) \geq 0$, and $[ry'']'_c \leq 0$; therefore $y(a) \geq 0$, $y'(a) \geq 0$, $y''(a) \geq 0$, and $[r(c)y''(a)]' \geq 0$. So $y$ satisfies Lemma 5.1 which states $y$, $y'$, $y''$, and $[ry'']'$ are strictly positive on $(a, \infty)$. Then $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, and $[r(t)z''(t)]' < 0$ for all $t$ in $(a_0, c)$. The interval is changed as a result of the transformation $x = a + c - t$. Notice that as $x$ increases $t$ must decrease; and $y(a) = z(c)$ and $y(a+c) = z(0)$.

LEMMA 5.2'. *If $y$ is a solution of (5.1) and if there is a $c$ in $(a_0, \infty)$ for which $y(c) \leq 0$, $y'(c) \geq 0$, $y''(c) \leq 0$, and $[ry'']'_c \geq 0$, then $y(x) < 0$, $y'(x) > 0$, $y''(x) < 0$, and $[ry'']'(x) > 0$ for all $x$ in $(a_0, c)$.*

PROOF. The proof follows directly from Lemma 5.2 if $y$ is replaced by $-y$. Note that $-y$ is a solution to (5.1) if $y$ is.

These four lemmas will be used in the proof of the following theorem.

THEOREM 5.3. *If $y$ is a nontrivial solution of (5.1) with $y(b) = y'(b) = 0$ and $y''(b) \neq 0$, where $b$ is in $(a_0, \infty)$, then in at least one of the intervals $(a_0, b)$ and $(b, \infty)$ all the functions $y$, $y'$, $y''$, and $[ry'']'$ are different from zero.*

PROOF. This situation is to be examined by cases.

Case I. Let $y''(b) > 0$. if $[ry'']_b' \geq 0$, then the conditions fit Lemma 5.1. Therefore all the functions $y$, $y'$, $y''$, and $[ry'']'$ are strictly positive on the interval $(b,\infty)$. If $[ry'']_b' < 0$, then the conditions fit Lemma 5.2. Then all four of the functions are nonzero on $(a_0, b)$.

Case II. Let $y''(b) < 0$. If $[ry''] \geq 0$, then the conditions fit Lemma 5.2'. Therefore the four functions are nonzero on $(a_0,b)$. If $[ry''] < 0$, then Lemma 5.1' may be applied and all four functions are strictly negative on $(b,\infty)$.

In all cases the functions $y$, $y'$, $y''$, and $[ry'']'$ are nonzero on either $(a_0,b)$ or $(b,\infty)$.

COROLLARY 5.4. *Let $y$ be a nontrivial solution to (5.1). If $y$ has a double zero at $b$, $a_0 < b$, then all other zeros of $y$ must be either entirely in $(a_0, b)$ or entirely in $(b, \infty)$.*

COROLLARY 5.5. *Let $y$ be a nontrivial solution of (5.1). If $y$ has a double zero at $b$ and another double zero at $c$, $a_0 < b < c$, then all other zeros of $y$ are restricted to $(b,c)$.*

COROLLARY 5.6. *If $y$ is a nontrivial solution of (5.1), then it can have at most two double zeros.*

THEOREM 5.7. *If $y$ is a nontrivial solution of (5.1) such that* $y(b) = y'(b) = y''(b) = 0$ for $b$ in $(a_0, \infty)$, *then $y$ has no other zeros on $(a_0, \infty)$.*

PROOF. if $[ry'']_b' \geq 0$, then by Lemma 5.1 $y$ is nonzero on $(b,\infty)$, further by Lemma 5.2' $y$ is nonzero on $(a_0,b)$. If $[ry'']_b' < 0$, then by Lemma 5.2 $y$ is nonzero on $(a_0,b)$, likewise by Lemma 5.1' $y$ is nonzero on $(b,\infty)$. Then in all cases $y(x) \neq 0$ for $x \neq b$ in $(a_0, \infty)$. If $[ry'']_b' = 0$, then $y$ would be the trivial solution.

COROLLARY 5.8. *Let* $y$ *be a nontrivial solution of* (5.1) *and let* $a$, $b$, *and* $c$ *be numbers such that* $a_0 < a < b < c$. *If* $y(a) = y(b) = y(c) = 0$, *then* $y'(b) \neq 0$.

THEOREM 5.9. *Equation* (5.1) *is in the class* F.

PROOF. Corollary 5.4 and Theorem 5.7 give the desired results.

## II. EXAMPLES

It is of interest to know there are equations of the type (1.1) in the class F that are not of the type (5.1). It will be sufficient to ~~examine examples of such equa~~tions.

The simplest such example is the equation

$$y^{(4)}(x) = 0, \tag{6.1}$$

which is obviously not of the type (5.1). All solutions to the equation are of the form

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3,$$

where the $c_i$'s are constants. Any solution, in which $c_4 \neq 0$, is a cubic polynomial. By the Fundamental Theorem of Algebra, a cubic polynomial has exactly three roots. Therefore every nontrivial solution to (6.1) has at most three zeros.

Using a result of Barrett [2], one may construct examples of equations in $F$ that have middle terms. However, examples constructed in this way may be transformed into equations of the form (5.1). Such transformations have also been used by Leighton and Nehari [4].

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